

A NOTE ON THE LIÉNARD-CHIPART CRITERION AND ROOTS OF SOME FAMILIES OF POLYNOMIALS

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ABSTRACT. We present some inequalities that provide different sufficient conditions for an univariate monic polynomial to be Hurwitz unstable. These are motivated by difficult control problems where direct application of the Liénard-Chipart criterion is not feasible. Hurwitz stability of some polynomials of degree five is also discussed. These results may be interpreted as stability results for some interval polynomials.

1. INTRODUCTION

In this note we'll say that a polynomial

$$p(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$$

is Hurwitz stable if all roots have negative real part. Otherwise, the polynomial will be said Hurwitz unstable.

When α_i is positive for $i = 1, \dots, n$, that is, all coefficients of $p(s)$ are positive, it is straight forward to prove that no real root can be strictly positive: Actually, it suffices to see that if $s > 0$ then $p(s) > 0$, but this observation also follows from Descartes' rule of signs. Furthermore if $p(s)$ has a root in $s = 0$ then $\alpha_n = 0$.

Hence, if we're trying to prove stability of a polynomial and if all coefficients α_i of $p(s)$ are strictly positive, we are left only to worry about the possibility of one of the complex roots to have positive real part. This still is a very complicated problem. For instance, bounds for roots like the ones derived from well-known ideas in complex analysis are not immediately useful since, a priori, all restrictions are given in absolute value (a non-trivial result that can be obtained using complex analysis is given by Routh's algorithm, see [2]).

The Liénard-Chipart stability criterion is a standard tool to understand the Hurwitz stability problem, which in turn has important consequences on the dynamics of some systems of differential equations. The theorem can be enunciated in the following way:

Theorem 1 (see [2], [4]). *Let $p(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$. A necessary and sufficient condition for all roots of $p(z)$ to have negative real parts is that*

$$\alpha_i > 0, \forall i = 1, 2, \dots, n \quad \text{and} \quad \Delta_2 > 0, \Delta_4 > 0, \dots$$

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where

$$\Delta_i = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \dots \\ 1 & \alpha_2 & \alpha_4 & \dots \\ 0 & \alpha_1 & \alpha_3 & \dots \\ 0 & 1 & \alpha_2 & \dots \\ 0 & 0 & \alpha_1 & \dots \\ \vdots & 0 & 1 & \\ \vdots & \vdots & & \\ 0 & 0 & \dots & \dots & \alpha_i \end{vmatrix}$$

with $\alpha_k = 0$ if $k > n$.

This result can be thought as a simplification of the Routh-Hurwitz theorem in which, if all coefficients of $p(s)$ are positive, stability can be guaranteed by checking that Δ_i is positive for $i = 1, \dots, n$. The Liénard-Chipart criterion can also be stated using Δ_i for i odd instead of even.

Although linearization together with the Liénard-Chipart criterion may reduce the dynamic stability problem to a calculation (see [3]), control problems in engineering can generate, in this manner, very complicated algebraic expressions that resist to simplifications, even by means of computational algebra (see for instance [5]). Numerical simulations are often the only solution available to study the dynamics, as we resort to sampling values for the coefficients. On the other hand generic conditions on the coefficients, like the aforementioned $\alpha_i > 0$, may be physically natural, easier to evaluate or even be chosen by design.

What we aim to do in this note is simplify a condition in the Liénard-Chipart criterion as much as possible, to provide more approachable and non-trivial necessary or sufficient conditions for the Hurwitz (in)stability of some polynomials. With this in mind, and assuming that all α_i are positive, it is worthwhile to study the sign of Δ_2 instead of Δ_3 , so the next term to study is Δ_4 , with the sign of Δ_6 being apparently much harder to understand.

To study the sign of Δ_4 we use a, to the best of our knowledge, new formula for Δ_4 which is presented in Theorem 2 in the next section.

2. THE MAIN RESULT

Theorem 2. *We have that*

$$\Delta_4 = -\alpha_2(\alpha_5 - \alpha_1\alpha_4)\Delta_2 - \alpha_4\Delta_2^2 - (\alpha_5 - \alpha_1\alpha_4)^2 - (\alpha_7 - \alpha_1\alpha_6)\Delta_2$$

where $\Delta_2 = \alpha_1\alpha_2 - \alpha_3$.

Proof. By definition

$$\Delta_4 = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \alpha_7 \\ 1 & \alpha_2 & \alpha_4 & \alpha_6 \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 \\ 0 & 1 & \alpha_2 & \alpha_4 \end{vmatrix}$$

Expansion by the first column leads to the expression

$$\begin{aligned} \Delta_4 = & \alpha_1\alpha_2\alpha_3\alpha_4 + 2\alpha_1\alpha_4\alpha_5 - \alpha_1\alpha_2^2\alpha_5 - \alpha_1^2\alpha_4^2 - \alpha_3^2\alpha_4 - \alpha_5^2 + \alpha_2\alpha_3\alpha_5 + \\ & + \alpha_1^2\alpha_2\alpha_6 - \alpha_1\alpha_3\alpha_6 - \alpha_1\alpha_2\alpha_7 + \alpha_3\alpha_7 \end{aligned}$$

that can be obtained by developing the formula given in the statement. \square

The alternative formula for Δ_4 in Theorem 2 was originally obtained in most part by the method described in [1] for degree five (where α_6 and α_7 are both zero).

This formula facilitates the determination of the sign of Δ_4 , as we'll see bellow, and can also be easily implemented to minimize computational cost and cumulative error of non-specialized software, being particularly useful to study polynomials of degree five.

3. SOME CONSEQUENCES FOR HURWITZ INSTABILITY

Corollary 1. *Let $p(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$, with $n \in \mathbb{N}$ greater than or equal to 5. Then a sufficient condition for $p(s)$ to be unstable is that*

$$\alpha_5 - \alpha_1 \alpha_4 \geq 0 \quad \text{and} \quad \alpha_7 - \alpha_1 \alpha_6 \geq 0$$

Proof. We can assume that $\alpha_i > 0$, for $i = 1, \dots, n$ and $\Delta_2 > 0$, since otherwise $p(s)$ is automatically unstable by the Liénard-Chipart criterion (Theorem 1). Therefore, by Theorem 2, $\Delta_4 \leq 0$ so $p(s)$ is Hurwitz unstable. \square

Note that the second inequality in the corollary is automatically satisfied if the degree of $p(s)$ is five.

Corollary 2. *If $\alpha_7 - \alpha_1 \alpha_6 \geq 0$ and n is greater than or equal to 5 the polynomial*

$$p(s) = s^n + \alpha_1 s^{n-1} + 2s^{n-2} + \alpha_3 s^{n-3} + s^{n-4} + \dots + \alpha_n$$

is unstable

Proof. Assume that $\Delta_2 > 0$. Since $\alpha_2 = 2$ and $\alpha_4 = 1$ we have that

$$\begin{aligned} \Delta_4 &= -2(\alpha_5 - \alpha_1 \alpha_4) \Delta_2 - \Delta_2^2 - (\alpha_5 - \alpha_1 \alpha_4)^2 - (\alpha_7 - \alpha_1 \alpha_6) \Delta_2 = \\ &= -[(\alpha_1 \alpha_2 - \alpha_3) + (\alpha_5 - \alpha_1 \alpha_4)]^2 - (\alpha_7 - \alpha_1 \alpha_6) \Delta_2 \leq 0 \end{aligned}$$

\square

Recall that if every α_i is strictly positive then all real roots must be strictly negative. Therefore in such a case and if $p(s)$ is as in Corollary 2 then $p(s)$ needs to have at least a pair of complex conjugated roots with positive real part. Also, if n is odd we can assure that there will be at least one negative real root.

Corollary 3. *Let $p(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$, with $n \in \mathbb{N}$ greater than or equal to 5. Then a sufficient condition for $p(s)$ to be unstable is that*

$$\alpha_2^2 - 4\alpha_4 \leq 0 \quad \text{and} \quad \alpha_7 - \alpha_1 \alpha_6 \geq 0$$

Proof. Note that if $\alpha_5 - \alpha_1 \alpha_4 = 0$ then $\Delta_4 < 0$ so there's nothing to do. Otherwise define

$$\Gamma := \frac{\Delta_4 + (\alpha_7 - \alpha_1 \alpha_6) \Delta_2}{(\alpha_5 - \alpha_1 \alpha_4)^2} = -\alpha_2 \frac{(\alpha_1 \alpha_2 - \alpha_3)}{(\alpha_5 - \alpha_1 \alpha_4)} - \alpha_4 \frac{(\alpha_1 \alpha_2 - \alpha_3)^2}{(\alpha_5 - \alpha_1 \alpha_4)^2} - 1$$

and note that $\Gamma < 0$ implies that $\Delta_4 < 0$. Let

$$\gamma := \frac{(\alpha_1 \alpha_2 - \alpha_3)}{(\alpha_5 - \alpha_1 \alpha_4)}$$

so $\Gamma = -\alpha_4 \gamma^2 - \alpha_2 \gamma - 1$.

As a function of γ , Γ is concave, since we can assume that $\alpha_4 > 0$. Then for $\Gamma(\gamma)$ to be positive for some γ we need to have that $\alpha_2^2 - 4\alpha_4 > 0$ and that

$$\frac{\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_4}}{-2\alpha_4} < \gamma < \frac{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_4}}{-2\alpha_4}$$

□

An easy consequence of the previous result is that, if $\alpha_7 - \alpha_1\alpha_6 \geq 0$,

$$p(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \alpha_3 s^{n-3} + s^{n-4} + \dots + \alpha_n$$

is unstable for every $0 < \alpha_2 \leq 2$ (compare to Corollary 2).

4. SOME CONSEQUENCES FOR HURWITZ STABILITY

A reformulation of the previous corollary gives a criteria for stability.

Corollary 4. *Let $p(s) = s^5 + \alpha_1 s^4 + \alpha_2 s^3 + \alpha_3 s^2 + \alpha_4 s + \alpha_5$. Assume that $\alpha_i > 0$ for $i = 1, \dots, 5$ and that $\Delta_2 > 0$. Then a necessary condition for $p(s)$ to be Hurwitz stable is that*

$$\alpha_2^2 - 4\alpha_4 > 0$$

Note that the hypotheses of the former corollary implies in particular that $(\alpha_5 - \alpha_1\alpha_4) < 0$, since otherwise $\Gamma(\gamma) < 0$ (see Corollary 1). Compare this to how the Liénard-Chipart shows that some conditions of the Routh-Hurwitz theorem are not independent. Our final result is still impractical as a tool to study problems like in [5], but is provided as an alternative to direct application of Theorem 1 or to the final inequality in Corollary 3.

Corollary 5. *Let $p(s) = s^5 + \alpha_1 s^4 + \alpha_2 s^3 + \alpha_3 s^2 + \alpha_4 s + \alpha_5$ where $\alpha_i > 0$ for $i = 1, 2, 3, 4, 5$. Assume that $\Delta_2 > 0$ and assume that*

$$\alpha_2^2 - 4\alpha_4 > 0$$

Then a sufficient condition for $p(s)$ to be stable is that

$$\frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_5 - \alpha_1\alpha_4} = \frac{-\alpha_2}{2\alpha_4}$$

Proof. Just note that if $\bar{\gamma}$ denotes the vertex of $\Gamma(\gamma)$ then $\bar{\gamma} := \frac{-\alpha_2}{2\alpha_4}$. Since $\Gamma(\gamma)$ has two distinct roots then $\Gamma(\gamma) = \Gamma(\bar{\gamma}) > 0$, therefore $\Delta_4 > 0$.

It's also possible to check directly that $\Delta_4 > 0$ by studying the sign of Γ . □

Finally, we note that the sufficient condition above is equivalent to

$$\Delta_2 = \alpha_3 - \frac{\alpha_2\alpha_5}{\alpha_4} \quad \text{and to} \quad \alpha_1 = \frac{2\alpha_3}{\alpha_2} - \frac{\alpha_5}{\alpha_4}$$

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